

CORIOLIS FORCE

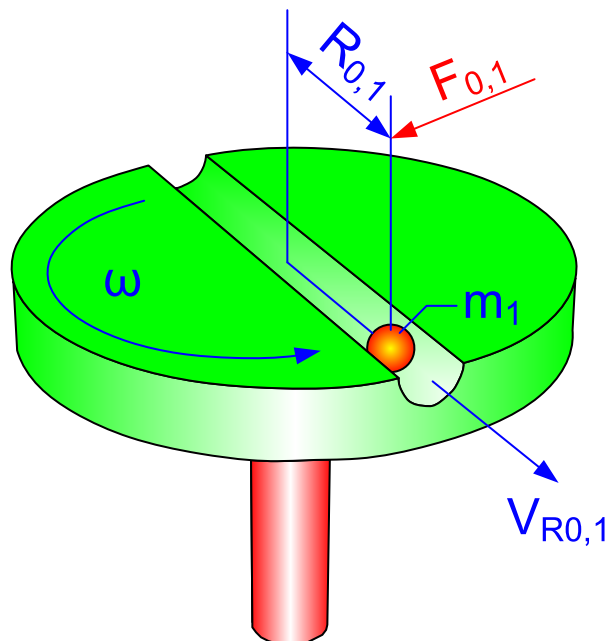
Abstract

Coriolis force is frequently deriving mystically and without proper explanation how equation (33) is derived only from equation (1), frequently inspiring strange ideas and inventions to students, mostly physically and mechanically impossible. The goal of this text is to detach phenomena that could be entirely handled by Classical Mechanics from ones that cannot be handled by it and to help to talented students to focus to ones that presently do not have proper explanations within the Classical Mechanics. This plain derivation of the Coriolis force in this paper should give to students a clear view to the effect – the purified view without any single mystification.

Although the Coriolis¹ is known for several centuries, its derivation is usually inappropriate and unclear – almost in the domain of dogma. Therefore it is decided to be derived this force by simple and plain set of equations based on basic equations of classical mechanics, just to be shown that there is nothing strange and unexplainable, as it is frequently speculating in various papers around the Internet. It is also interesting to be noticed that angular acceleration is able to dismantle acting force between two point preserving its direction – therefore we have a force whose one end is directed to the center of rotation and the another one is directed to the mass moving in the groove on the rotating platform, although we intuitively feel that the force should be directed between the groove's wall and the mass! Actually, this force is transferred trough the rigid groove's rim directly to the center of rotation – although explanation is simple, the mathematical formalism is a bit confusing here.

By definition, Coriolis force is the force that acts to the mass moving crosswise the groove on the rotating platform and it is perpendicular to the groove itself:

Fig. 1



¹ Gustave Coriolis, 1792 – 1843

Connection between Coriolis force and Coriolis acceleration is given by the following well known Newton formula:

$$\vec{F}_{0,1} = m_1 \cdot \vec{a}_{0,1} \quad (1)$$

Whereas $\vec{F}_{0,1}$ is the force that acts perpendicularly to direction of radius vector and it is created by the body m_1 exposed to acceleration while moving along radius $R_{0,1}$.

While it is quite clear that point 1 is settled on the body m_1 that is moving along the radius, the second point remains in the domain of imagination as virtual point somehow dismantled from the its origin to the center of rotation. In reality the situation is clear and the force is mechanically transferred trough rigid rim of the groove right to the center of rotation, but it is not obvious from the vector equations whose denote only that the force is displaced between two distinct points.

Relativists often use Coriolis force's equation for comparison of relativity equations with the Maxwell² ones, although this equation is entirely derived from the third Newton³ law (1) and Maxwell equations do not contain any repetition of this kind at all and therefore such relation is false.

Derivation of Coriolis force will be derived in both Cartesian⁴ and Polar coordinate systems to clarify any doubt about existence of any hidden and mystique property.

DERIVATION OF CORIOLIS ACCELARATION IN CARTESIAN COORDIANTES

Radius vector of the m_1 particle moving is:

$$\vec{R}_{0,1} = [x \ y \ 0] \quad (2)$$

So, the Coriolis force is given as the perpendicular projection of the force acting to the mass m_1 on the radius vector $\vec{R}_{0,1}$:

$$\vec{a}_{0,1} = \vec{a}_\tau + \vec{a}_\perp \quad (3)$$

Total acceleration is sum of perpendicular and tangential accelerations. Coriolis acceleration is the perpendicular one and therefore according above formula it is:

$$\vec{a}_\perp = \vec{a}_{\text{Coriolis}} = \vec{a}_{0,1} - \vec{a}_\tau = \vec{a}_{0,1} - (\vec{a}_{0,1} \cdot \hat{R}_{0,1}) \cdot \hat{R}_{0,1} = \hat{R}_{0,1} \times \left(\frac{d^2 \vec{R}_{0,1}}{dt^2} \times \hat{R}_{0,1} \right) \quad (4)$$

Thereby the relevant components of the radius vector are:

$$x = d_{R_{0,1}} \cdot \cos(\theta) \quad (5)$$

² James Maxwell, 1831 - 1879

³ Isaac Newton, 1643-1727

⁴ Renatus Cartesius (i.e. René Descartes), 1596-1650

And:

$$y = d_{R_{0,1}} \cdot \sin(\theta) \quad (6)$$

Total acceleration is given as second time derivative of radius vector (2) or second time derivative of radius vector's non-trivial components (5) and (6):

$$\frac{d^2x}{dt^2} = \ddot{x} = \frac{d^2}{dt^2} (d_{R_{0,1}} \cdot \cos(\theta)) \quad (7)$$

And:

$$\frac{d^2y}{dt^2} = \ddot{y} = \frac{d^2}{dt^2} (d_{R_{0,1}} \cdot \sin(\theta)) \quad (8)$$

Symbolic solutions of above derivatives are:

$$\frac{d^2x}{dt^2} = \ddot{x} = \cos(\theta) \cdot (a_{R_{0,1}} - R_{0,1} \cdot \omega^2) - \sin(\theta) \cdot (2 \cdot \omega \cdot v_{R_{0,1}} + R_{0,1} \cdot \alpha) \quad (9)$$

And:

$$\frac{d^2y}{dt^2} = \ddot{y} = \cos(\theta) \cdot (2 \cdot \omega \cdot v_{R_{0,1}} + R_{0,1} \cdot \alpha) + \sin(\theta) \cdot (a_{R_{0,1}} - R_{0,1} \cdot \omega^2) \quad (10)$$

Whereas:

- $a_{R_{0,1}}$ = acceleration in regards to the radius vector itself,
- $R_{0,1}$ = distance from the center of the rotation, $R_{0,1} = |\vec{R}_{0,1}|$
- $v_{R_{0,1}}$ = velocity in regards to the radius vector, $v_{R_{0,1}} = \frac{R_{0,1}}{dt} = \dot{R}_{0,1}$
- ω = angular velocity, $\omega = \frac{d\theta}{dt} = \dot{\theta}$
- α = angular acceleration, $\alpha = \frac{d^2\theta}{dt^2} = \ddot{\theta}$

According (4) we have:

$$\vec{a}_{\text{Coriolis}} = (2 \cdot \omega \cdot v_{R_{0,1}} + R_{0,1} \cdot \alpha) \cdot [-\sin(\theta) \quad \cos(\theta) \quad 0] \quad (11)$$

So, the Coriolis force is stretched between the mass and the center of the rotation, but the force is perpendicular to the radius and therefore it is the case of dismantled force and similar effect exists in electromagnetic interaction and therefore the magnetic field is perpendicular to the magnetic force.

More convenient form is:

$$\vec{a}_{\text{Coriolis}} = (2 \cdot \omega \cdot v_{R_{0,1}} + R_{0,1} \cdot \alpha) \cdot \sin(\theta) \cdot \hat{i}_x + (2 \cdot \omega \cdot v_{R_{0,1}} + R_{0,1} \cdot \alpha) \cdot \cos(\theta) \cdot \hat{i}_y \quad (12)$$

Whereas $\hat{i}_x$ is ort vector of x axe and $\hat{i}_y$ is ort vector of y axe of the Cartesian coordinate system.

Absolute value or magnitude of the Coriolis acceleration is:

$$a_{\text{Coriolis}} = 2 \cdot \omega \cdot v_{R_{0,1}} + R_{0,1} \cdot \alpha \quad (13)$$

In accordance with (1) Coriolis force is:

$$F_{\text{Coriolis}} = 2 \cdot m_1 \cdot \omega \cdot v_{R_{0,1}} + m_1 \cdot R_{0,1} \cdot \alpha \quad (14)$$

Whereas ω is angular velocity, α is angular acceleration, $d_{R_{0,1}}$ is distance from the center of rotation or absolute radius, $v_{R_{0,1}}$ is velocity across the radius, i.e. velocity of the mass moving in the groove in respect to the center of rotation.

Usual form of Coriolis formula does not assume that there is angular acceleration, and therefore for steady rotation it is:

$$a_{\text{Coriolis}} \approx 2 \cdot \omega \cdot v_{R_{0,1}} \quad (15)$$

It is interesting to be noticed that centrifugal force varies with radius and that this is not case with the Coriolis force:

$$a_{\text{Centrifugal}} = \omega^2 \cdot R_{0,1} \quad (16)$$

Coriolis had discovered his force in a study which should determine mechanical cause of systematic target missing of long-range artillery.

DERIVATION OF CORIOLIS ACCELARATION IN POLAR COORDIANTES

According definition formula for Coriolis acceleration it is:

$$\vec{a}_{\text{Coriolis}} = \hat{R}_{0,1} \times \left(\frac{d^2 \vec{R}_{0,1}}{dt^2} \times \hat{R}_{0,1} \right) \quad (17)$$

Whereas:

$$\vec{R}_{0,1} = d_{R_{0,1}} \cdot \hat{R}_{0,1} \quad (18)$$

The derivation of the ort (unit) vectors must be firstly determined before Coriolis formula is applied:

$$\frac{d|\vec{R}_{0,1}|}{dt} = \hat{R}_{0,1} \cdot \frac{d\vec{R}_{0,1}}{dt} = \hat{R}_{0,1} \cdot \left(v_{R_{0,1}} \cdot \hat{R}_{0,1} + v_{R_{0,1}} \cdot \frac{d\hat{R}_{0,1}}{dt} \right) \quad (19)$$

It is necessary to be derived formula of angular velocity of the radius vector for further derivation Coriolis force. Let us start from the definition of the vectors product:

$$|\vec{R} \times \vec{K}| = |\vec{R}| \cdot |\vec{K}| \cdot \sin(\theta) \quad (20)$$

When vector $\vec{K}$ is replaced by $\vec{R} + d\vec{R}$ we have:

$$|\vec{R} \times (\vec{R} + d\vec{R})| = |\vec{R}| \cdot |\vec{R} + d\vec{R}| \cdot \sin(d\theta) \quad (21)$$

$\Rightarrow$

$$|\vec{R} \times d\vec{R}| = \vec{R}^2 \cdot d\theta \quad (22)$$

While vectors $\vec{R} \times d\vec{R}$ and $d\theta$ are parallel ones, we have:

$$\vec{R} \times d\vec{R} = \vec{R}^2 \cdot d\vec{\theta} \quad (23)$$

Or, when above formula is divided by dt, it is obtained:

$$\vec{\omega}_R = \frac{1}{\vec{R}^2} \cdot \left(\vec{R} \times \frac{d\vec{R}}{dt} \right) = \frac{1}{|\vec{R}|} \cdot \left(\hat{R} \times \frac{d\vec{R}}{dt} \right) \quad (24)$$

Finally we have:

$$\frac{d\hat{R}_{0,1}}{dt} = \frac{\hat{R}_{0,1} \times \left(\frac{d\vec{R}_{0,1}}{dt} \times \hat{R}_{0,1} \right)}{|\vec{R}_{0,1}|} = \vec{\omega}_{R_{0,1}} \times \hat{r} = \vec{\omega}_{R_{0,1}} \times \hat{R}_{0,1} \quad (25)$$

Therefore is:

$$\frac{d\vec{R}_{0,1}}{dt} = v_{R_{0,1}} \cdot \hat{R}_{0,1} + \left(\vec{\omega}_{R_{0,1}} \times (d_{R_{0,1}} \cdot \hat{r}) \right) \quad (26)$$

$\Rightarrow$

$$\frac{d^2\vec{R}_{0,1}}{dt^2} = \vec{a}_{R_{0,1}} + 2 \cdot \left(\vec{\omega}_{R_{0,1}} \times \vec{v}_{R_{0,1}} \right) + \left(\vec{\alpha}_{R_{0,1}} \times \vec{R}_{0,1} \right) + \vec{\omega}_{R_{0,1}} \cdot \left(\vec{\omega}_{R_{0,1}} \cdot \vec{R}_{0,1} \right) - \vec{\omega}_{R_{0,1}}^2 \cdot \vec{R}_{0,1} \quad (27)$$

According (17) is:

$$\vec{a}_{Coriolis} = \frac{d^2\vec{R}_{0,1}}{dt^2} - \left(\frac{d^2\vec{R}_{0,1}}{dt^2} \cdot \hat{R}_{0,1} \right) \cdot \hat{R}_{0,1} \quad (28)$$

It is convenient now to be found following term:

$$\frac{d^2\vec{R}_{0,1}}{dt^2} \cdot \hat{R}_{0,1} = a_{R_{0,1}} + \left(\vec{\omega}_{R_{0,1}} \cdot \hat{R}_{0,1} \right)^2 \cdot |\vec{R}_{0,1}| - \vec{\omega}_{R_{0,1}}^2 \cdot |\vec{R}_{0,1}| \quad (29)$$

$\Rightarrow$

$$\left(\frac{d^2 \vec{R}_{0,1}}{dt^2} \cdot \hat{R}_{0,1} \right) \cdot \hat{R}_{0,1} = \vec{a}_{R_{0,1}} + \left(\vec{\omega}_{R_{0,1}} \cdot \hat{R}_{0,1} \right)^2 \cdot \vec{R}_{0,1} - \vec{\omega}_{R_{0,1}}^2 \cdot \vec{R}_{0,1} \quad (30)$$

Following equation for Coriolis force is derived in accordance to formulas (27), (28) and (30):

$$\vec{a}_{\text{Coriolis}} = 2 \cdot \left(\vec{\omega}_{R_{0,1}} \times \vec{v}_{R_{0,1}} \right) + \vec{\alpha}_{R_{0,1}} \times \vec{R}_{0,1} + \vec{\omega}_{R_{0,1}} \cdot \left(\vec{\omega}_{R_{0,1}} \cdot \vec{R}_{0,1} \right) - \left(\vec{\omega}_{R_{0,1}} \cdot \hat{R}_{0,1} \right)^2 \cdot \vec{R}_{0,1} \quad (31)$$

While $\vec{\omega}_{R_{0,1}}$ is perpendicular to $\vec{R}_{0,1}$, i.e. $\vec{\omega}_{R_{0,1}} \perp \vec{R}_{0,1}$ we have:

$$\vec{a}_{\text{Coriolis}} = 2 \cdot \left(\vec{\omega}_{R_{0,1}} \times \vec{v}_{R_{0,1}} \right) + \vec{\alpha}_{R_{0,1}} \times \vec{R}_{0,1} \quad (32)$$

Therefore Coriolis force is:

$$\vec{F}_{\text{Coriolis}} = 2 \cdot m_1 \cdot \left(\vec{\omega}_{R_{0,1}} \times \vec{v}_{R_{0,1}} \right) + m_1 \cdot \vec{\alpha}_{R_{0,1}} \times \vec{R}_{0,1} \quad (33)$$

Thus we proved that we have obtained the same result in both Cartesian (11) and polar coordinate (32) systems by usage of Newton's formula (1) only. Therefore Coriolis force is just a very special case of a Newton's inertial force.

FOUCAULT'S PENDULUM AND SINK WATER VORTEX

There are two interesting phenomena related to Coriolis force – Foucault⁵ pendulum and Sink Water Vortex. Although Foucault pendulum matches at least the within the order of magnitude of the theoretical value obtained by the pendulum exposed to Coriolis force (about 20% is error which is really good), the case with the sink vortex is much more complicated. This vortex definitely depends on the Earth's rotation while on south hemisphere it rotates in opposite direction then on the north one. However, the Coriolis force in the sink is really negligible and it certainly cannot cause so vigorous rotation of the vortex! Only hope is that additional term with acceleration can handle this rotation, but – this is not case. However, this term is able to further harmonize Foucault pendulum with theory of classical mechanics, but sink's vortex remains unexplained phenomena within classical mechanics.

CONCLUSION

Coriolis force can be handled by the classical mechanics only, entirely relied on the Newton equation (1). However, there is small chance that sink vortex is caused by the true gravitational induction, which means that this effect is beyond the scope of aforesaid equation (1), which gives us a hope that technical utilization of anti-gravitation is a goal that is still attainable to our technical civilization.

Author:
Dipl.-Ing. Andrija Radović
<http://www.andrijar.com>

⁵ Léon Foucault, 1819-1868