Abstract

This text is dedicated to people that do not have adequate university education and are not familiar with the formalism and abstractions exposed in the books about Mechanics that deal with gyroscopes. For adoption of such approach the certain introduction and training of Higher Mechanics are necessary. But, the effect can be described only by elementary equations of particle's motion. For understating of such approach only the high school knowledge of mechanics formulas, elementary geometry and the fundamentals of the vector and infinitesimal calculus are necessary.
The explanation of the gyroscopic effect is based on the queue of the simple equations that finally bring us to the formula of the effect.

Gyroscopic effect

Lets imagine the tiny contour with the radius $\mathbf{r_0}$ that rotates around its axe – **r** axe and also around the axe normal to **r** axe of rotation – **z** axe. The circular velocities around axes **r** and **z** are denoted with the $\boldsymbol{\omega_r}$ and $\boldsymbol{\omega_z}$ respectively.
Then the position of every particle is determined by the following equations:

$$x = r_0 \cdot SIN(\omega_r \cdot t) \cdot COS(\omega_z \cdot t) \tag{1}$$

$$y = r_0 \cdot SIN(\omega_r \cdot t) \cdot SIN(\omega_z \cdot t) \tag{2}$$

$$z = r_0 \cdot COS(\omega_r \cdot t) \tag{3}$$

The acceleration is obtained as the second time derivation of coordinates:

$$\ddot{x} = -r_0 \cdot \left(2 \cdot \omega_r \cdot \omega_z \cdot COS(\omega_r \cdot t) \cdot SIN(\omega_z \cdot t) + \left(\omega_r^{\,2} + \omega_z^{\,2}\right) \cdot SIN(\omega_r \cdot t) \cdot COS(\omega_z \cdot t)\right) \tag{4}$$

$$\ddot{y} = r_0 \cdot \left(2 \cdot \omega_r \cdot \omega_z \cdot COS(\omega_r \cdot t) \cdot COS(\omega_z \cdot t) - \left(\omega_r^{\,2} + \omega_z^{\,2}\right) \cdot SIN(\omega_r \cdot t) \cdot SIN(\omega_z \cdot t)\right) \tag{5}$$

$$\ddot{z} = -r_0 \cdot \omega_r^{\,2} \cdot COS(\omega_r \cdot t) \tag{6}$$

Hence it is:

$$\omega_r = \frac{\theta}{t} \tag{7}$$

And

$$\omega_z = \frac{\alpha}{t} \tag{8}$$

Angles $\boldsymbol{\theta}$ and $\boldsymbol{\alpha}$ describes angular shift of axes **r** and **z** from the plane in which the contour was at the beginning of observation. Consequently we have:

$$\ddot{x} = -r_0 \cdot \left(2 \cdot \omega_r \cdot \omega_z \cdot COS(\theta) \cdot SIN(\alpha) + \left(\omega_r^{\,2} + \omega_z^{\,2}\right) \cdot SIN(\theta) \cdot COS(\alpha)\right) \tag{9}$$

$$\ddot{y} = r_0 \cdot \left(2 \cdot \omega_r \cdot \omega_z \cdot \mathrm{COS}(\theta) \cdot \mathrm{COS}(\alpha) - \left(\omega_r^{\ 2} + \omega_z^{\ 2}\right) \cdot \mathrm{SIN}(\theta) \cdot \mathrm{SIN}(\alpha)\right) \tag{10}$$

$$\ddot{z} = -r_0 \cdot \omega_r^{\ 2} \cdot \mathrm{COS}(\theta) \tag{11}$$

Every moment could be choose and the most suitable one is when contour is in position equal to the starting position and it is the position when the **α = 0**:

$$a_x\big|_{\alpha=0} = \ddot{x}\big|_{\alpha=0} = -r_0 \cdot \left(\omega_r^{\ 2} + \omega_z^{\ 2}\right) \cdot \mathrm{SIN}(\theta) \tag{12}$$

$$a_y\big|_{\alpha=0} = \ddot{y}\big|_{\alpha=0} = 2 \cdot r_0 \cdot \omega_r \cdot \omega_z \cdot \mathrm{COS}(\theta) \tag{13}$$

$$a_z\big|_{\alpha=0} = \ddot{z}\big|_{\alpha=0} = -r_0 \cdot \omega_r^{\ 2} \cdot \mathrm{COS}(\theta) \tag{14}$$

Vector $\vec{a}$ can be written as:

$$\vec{a} = \left\lfloor a_x \quad a_y \quad a_z \right\rfloor \tag{15}$$

The basic formula for the particle motion is given by:

$$\vec{F} = m \cdot \vec{a} \tag{16}$$

And while the next connection between force and moment of force also matters:

$$\vec{M} = \vec{r} \times \vec{F} = m \cdot \vec{r} \times \vec{a} \tag{17}$$

Finally we have:

$$d\vec{M} = dm \cdot \vec{r} \times \vec{a} \tag{18}$$

For the tiny contour the following formula is valid:

$$m' = \frac{dm}{d\ell} \tag{19}$$

Where **m'** is the mass across the unity of length. Further we have:

$$m = \int_0^{\ell} m' \cdot d\ell = m' \cdot 2 \cdot \pi \cdot r_0 \tag{20}$$

$\Rightarrow$

$$m' = \frac{m}{2 \cdot \pi \cdot r_0} \tag{21}$$

The definition of radian is:

$$\ell = r_0 \cdot \theta \tag{22}$$

$\Rightarrow$

$$dm = m' \cdot d\ell = \frac{m}{2 \cdot \pi \cdot r_0} \cdot r_0 \cdot d\theta \tag{23}$$

$\Rightarrow$

$$dm = \frac{m}{2 \cdot \pi} \cdot d\theta \tag{24}$$

Finally we have:

$$\vec{M} = \frac{m}{2 \cdot \pi} \cdot \int_0^{2 \cdot \pi} \vec{r}\Big|_{\alpha=0} \times \vec{a}\Big|_{\alpha=0} \cdot d\theta \tag{25}$$

Whereas:

$$\vec{r}\Big|_{\alpha=0} \times \vec{a}\Big|_{\alpha=0} = \begin{bmatrix} -2 \cdot r_0{}^2 \cdot \omega_r \cdot \omega_z \cdot COS(\theta)^2 \\ -r_0{}^2 \cdot \omega_z{}^2 \cdot SIN(\theta) \cdot COS(\theta) \\ 2 \cdot r_0{}^2 \cdot \omega_r \cdot \omega_z \cdot SIN(\theta) \cdot COS(\theta) \end{bmatrix}^T \tag{26}$$

When the integral (25) is solved than is obtained:

$$\vec{M} = \begin{bmatrix} -m \cdot r_0{}^2 \cdot \omega_r \cdot \omega_z & 0 & 0 \end{bmatrix} \tag{27}$$

Inertia of contour is calculating by the following formula:

$$I = \int_0^m r^2 \cdot dm = r_0{}^2 \cdot \int_0^m dm = m \cdot r_0{}^2 \tag{28}$$

Hence it is obtained:

$$\vec{M} = \begin{bmatrix} -I \cdot \omega_r \cdot \omega_z & 0 & 0 \end{bmatrix} \tag{29}$$

While these are in validation:

$$\vec{\omega}_r = \begin{bmatrix} 0 & \omega_r & 0 \end{bmatrix} \tag{30}$$

And

$$\vec{\omega}_z = \begin{bmatrix} 0 & 0 & \omega_z \end{bmatrix} \tag{31}$$

$\Rightarrow$

$$\vec{\omega}_r \times \vec{\omega}_z = \begin{bmatrix} \omega_r \cdot \omega_z & 0 & 0 \end{bmatrix} \tag{32}$$

Thus we conclude that the following formula is valid:

$$\vec{M} = I \cdot \vec{\omega}_z \times \vec{\omega}_r \tag{33}$$

By all these is demonstrated that dynamics of the material point is substantially enough for derivation of Gyroscopic effect formula and that all energy that produce the effect is accumulated in kinetic energy of rotating mass and consequently the device is not able to produce constant force in one direction. The force could be produced only by process that incorporates derivatives with orders higher then the second.

Author: Andrija S. Radović
E-mail: andrijar@eunet.yu