STUDENT FRIENDLY DERIVATION OF CORIOLIS FORCE

ABSTRACT

In the text below Coriolis force will be derived solely relying on the mathematical vectors' identities without need to visualize or conceive any physical abstraction at all, just successive application of mathematics will be used to derive this mechanical effect. This application of strict mathematical identities is suitable for students to understand the effect and the physical and mathematical reasons for its occurrence in real world, outside of abstract mathematical formulas. There is expectation in Classical Mechanics that all formula can be derived relaying only on two basic formulas $d\vec{F}_{1,2} = dm_1 \cdot \vec{a}_{1,2}$ and $d^2\vec{F}_{1,2} = \gamma \cdot \frac{dm_1 \cdot dm_2}{\vec{r}_{1,2}^{\,2}} \cdot \hat{r}_{1,2}$, therefore Coriolis force is not an exception at all, as it is conceivably shown in the text below. This effect is very important for understanding of turbulence and Turbo Effect in fluid dynamics.

For the correct derivation of the Coriolis force is necessary to be found following formula for the time derivations of the absolute value of a vector:

$$\frac{d}{dt}|\vec{\rho}| = \frac{d}{dt}\sqrt{\vec{\rho}^2} = \frac{\vec{\rho} \cdot \dot{\vec{\rho}}}{\sqrt{\vec{\rho}^2}} = \frac{\vec{\rho} \cdot \dot{\vec{\rho}}}{|\vec{\rho}|} = \dot{\vec{\rho}} \cdot \hat{\rho} \tag{1}$$

And also a formula of time derivation of the corresponding ort vector should be found:

$$\frac{d}{dt}\hat{\rho} = \frac{d}{dt}\frac{\vec{\rho}}{|\vec{\rho}|} = \frac{\dot{\vec{\rho}} \cdot |\vec{\rho}| - \vec{\rho} \cdot \left(\dot{\vec{\rho}} \cdot \hat{\rho}\right)}{|\vec{\rho}|^2} = \frac{\dot{\vec{\rho}} - \hat{\rho} \cdot \left(\dot{\vec{\rho}} \cdot \hat{\rho}\right)}{|\vec{\rho}|} = \frac{\hat{\rho} \times \left(\dot{\vec{\rho}} \times \hat{\rho}\right)}{|\vec{\rho}|} = \frac{\left(\vec{\rho} \times \dot{\vec{\rho}}\right)}{\vec{\rho}^2} \times \hat{\rho} = \vec{\omega}_C \times \hat{\rho} \tag{2}$$

Coriolis velocity $\vec{v}_C$ is the velocity of approaching to axis of rotation by its definition:

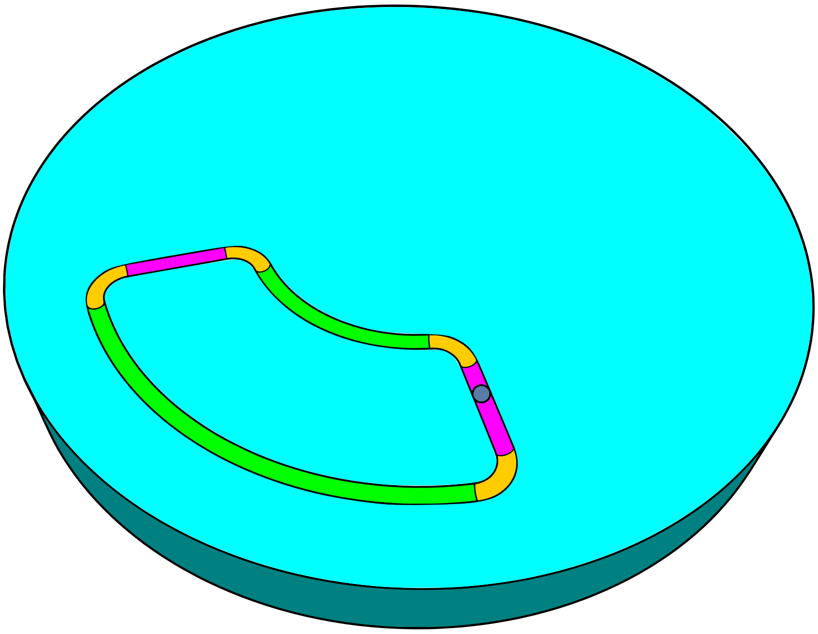
$$\vec{v}_C = \dot{\rho} \cdot \hat{\rho} = v_C \cdot \vec{\rho} \tag{3}$$

And:

$$\vec{R} = \rho \cdot \hat{\rho} \tag{4}$$

It is important here to be noticed that v_c is not the velocity in the respect to the rotating platform as it is frequently citied in the available literature from the domain of the Classical Mechanics, but it is velocity in the respect of axis of rotation. The following figure depicts a groove with a grey ball moving inside it with the constant speed v_0 on the rotating bluish disk. The groove contains green, purple and orange sections of the path and the Coriolis force appears only on trajectory in the purple sections of the groove because only in this section relative velocity to the respect of the center of rotation is not trivial, i.e. different than zero as it is case with green sections:

Fig. 1



Otherwise the Coriolis force would be twice stronger than the Centrifugal force on the greenish parts of the trajectory and would act in the opposite direction than the centrifugal force which is duly impossible - on these sections only centrifugal force acts without suitable circumstances for appearance of Coriolis force. Therefore Coriolis velocity $\vec{v}_C$ is undoubtedly the velocity of the body in respect to the axis of rotation only. Most misunderstanding of the Coriolis' effect was based on the false grasp of the Coriolis' velocity $\vec{v}_C$ which is somehow wrongly defined in the most of the textbooks of Classical Mechanics as the velocity in respect to the rotational disk's plane.

According (2) is:

$$\dot{\vec{R}} = \dot{\rho} \cdot \hat{\rho} + \rho \cdot \vec{\omega}_C \times \hat{\rho} \tag{5}$$

According (2) too:

$$\ddot{\vec{R}} = \ddot{\rho} \cdot \hat{\rho} + \dot{\rho} \cdot \vec{\omega}_C \times \hat{\rho} + \dot{\rho} \cdot \vec{\omega}_C \times \hat{\rho} + \rho \cdot \dot{\vec{\omega}}_C \times \hat{\rho} + \rho \cdot \vec{\omega}_C \times (\vec{\omega}_C \times \hat{\rho}) \tag{6}$$

According (3) is:

$$\ddot{\vec{R}} = \dot{\vec{v}}_C + 2 \cdot \vec{\omega}_C \times \vec{v}_C + \dot{\vec{\omega}}_C \times \vec{v}_C + \vec{\omega}_C \times (\vec{\omega}_C \times \vec{\rho}) \tag{7}$$

$\Rightarrow$

$$\ddot{\vec{R}} = \dot{\vec{v}}_C + 2 \cdot \vec{\omega}_C \times \vec{v}_C + \dot{\vec{\omega}}_C \times \vec{v}_C + \vec{\omega}_C \cdot (\vec{\omega}_C \cdot \vec{\rho}) - \vec{\omega}_C{}^2 \cdot \vec{\rho} \tag{8}$$

It is also $\vec{\omega}_C \perp \vec{\rho} \Rightarrow \vec{\omega}_C \cdot \vec{\rho} = 0$, so following formula is obtained:

$$\ddot{\vec{R}} = \dot{\vec{v}}_C - \vec{\omega}_C{}^2 \cdot \vec{\rho} + 2 \cdot \vec{\omega}_C \times \vec{v}_C + \dot{\vec{\omega}}_C \times \vec{v}_C \tag{9}$$

$\Rightarrow$

$$\vec{F} = m \cdot \vec{a} = m \cdot \ddot{\vec{R}} = \overbrace{m \cdot \dot{\vec{v}}_C}^{\text{Radial f.}} + \overbrace{m \cdot \dot{\vec{\omega}}_C \times \vec{v}_C}^{\text{Angular force}} - \overbrace{m \cdot \vec{\omega}_C{}^2 \cdot \vec{\rho}}^{\text{Centrifugal force}} + \overbrace{2 \cdot m \cdot \vec{\omega}_C \times \vec{v}_C}^{\text{Coriolis force}} \tag{10}$$

Or:

$$\vec{F} = \overbrace{m \cdot \ddot{\rho} \cdot \hat{\rho}}^{\text{Radial force}} + \overbrace{m \cdot \rho \cdot \dot{\vec{\omega}}_C \times \hat{\rho}}^{\text{Angular force}} - \overbrace{m \cdot \vec{\omega}_C{}^2 \cdot \hat{\rho}}^{\text{Centrifugal force}} + \overbrace{2 \cdot m \cdot \dot{\rho} \cdot \vec{\omega}_C \times \hat{\rho}}^{\text{Coriolis force}} \tag{11}$$

The first term denotes force caused by acceleration of the body in respect to the axe of rotation; the second term denotes force cause by angular acceleration of the platform, the third term denotes centrifugal force that appears when body runs on arc with constant distant from the axe of rotation (greenish section on above figure) and finally the fourth term denotes precious Coriolis force which appears only when the body has steady velocity on the line perpendicular to the axe of rotation. So, v_C is the velocity of going away from the axe of rotation and not any velocity of the body in respect to the platform.

Above formula clearly states that there is no Coriolis force on the greenish parts of the trajectory depicted on the Fig. 1 because the Coriolis velocity is equal to zero due to the constant radius of these paths making the term dedicated to the Coriolis force trivial.

According both (5) and (6) the work of Coriolis force is:

$$dA = \vec{F} \cdot \dot{\vec{R}} \cdot dt = m \cdot \left(\dot{\rho} \cdot \ddot{\rho} + \vec{\omega}_C{}^2 \cdot \rho \cdot \dot{\rho} + \left(\dot{\vec{\omega}}_C \cdot \vec{\omega}_C \right) \cdot \rho^2 \right) \tag{12}$$

Derivation of the work done by Coriolis force is:

$$\dot{\vec{R}} = \dot{\rho} \cdot \hat{\rho} + \rho \cdot \vec{\omega}_C \times \hat{\rho} \tag{13}$$

$\Rightarrow$

$$\ddot{\vec{R}} = \ddot{\rho} \cdot \hat{\rho} + 2 \cdot \dot{\rho} \cdot \vec{\omega}_C \times \hat{\rho} + \rho \cdot \dot{\vec{\omega}}_C \times \hat{\rho} + \rho \cdot \vec{\omega}_C \times \left(\vec{\omega}_C \times \hat{\rho} \right) \tag{14}$$

$\Rightarrow$

$$dA = \vec{F} \cdot \dot{\vec{R}} \cdot dt = m \cdot \ddot{\vec{R}} \cdot \dot{\vec{R}} \cdot dt \tag{15}$$

$\Rightarrow$

$$dA = m \cdot \left(\ddot{\rho} \cdot \hat{\rho} + 2 \cdot \dot{\rho} \cdot \vec{\omega}_C \times \hat{\rho} + \rho \cdot \dot{\vec{\omega}}_C \times \hat{\rho} + \rho \cdot \vec{\omega}_C \times \left(\vec{\omega}_C \times \hat{\rho} \right) \right) \cdot \left(\dot{\rho} \cdot \hat{\rho} + \rho \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) \right) \tag{16}$$

$\Rightarrow$

$$dA = m \cdot \left(\ddot{\rho} \cdot \hat{\rho} + 2 \cdot \dot{\rho} \cdot \vec{\omega}_C \times \hat{\rho} + \rho \cdot \dot{\vec{\omega}}_C \times \hat{\rho} - \rho \cdot \vec{\omega}_C{}^2 \cdot \hat{\rho} \right) \cdot \left(\dot{\rho} \cdot \hat{\rho} + \rho \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) \right) \tag{17}$$

$\Rightarrow$

$$dA = m \cdot \left(\ddot{\rho} \cdot \hat{\rho} - \rho \cdot \vec{\omega}_C{}^2 \cdot \hat{\rho} + 2 \cdot \dot{\rho} \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) + \rho \cdot \left(\dot{\vec{\omega}}_C \times \hat{\rho} \right) \right) \cdot \left(\dot{\rho} \cdot \hat{\rho} + \rho \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) \right) \tag{18}$$

$\Rightarrow$

$$dA = m \cdot \left(\ddot{\rho} \cdot \dot{\rho} - \rho \cdot \vec{\omega}_C{}^2 \cdot \dot{\rho} + 2 \cdot \dot{\rho} \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) \cdot \rho \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) + \rho \cdot \left(\dot{\vec{\omega}}_C \times \hat{\rho} \right) \cdot \rho \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) \right) \tag{19}$$

$\Rightarrow$

$$dA = m \cdot \left(\ddot{\rho} \cdot \dot{\rho} - \vec{\omega}_C{}^2 \cdot \rho \cdot \dot{\rho} + 2 \cdot \rho \cdot \dot{\rho} \cdot \left(\vec{\omega}_C \times \hat{\rho} \right)^2 + \rho^2 \cdot \left(\dot{\vec{\omega}}_C \times \hat{\rho} \right) \cdot \left(\vec{\omega}_C \times \hat{\rho} \right) \right) \tag{20}$$

$\Rightarrow$

$$dA = m \cdot \left(\dot{v}_C \cdot v_C - \vec{\omega}_C{}^2 \cdot \rho \cdot v_C + 2 \cdot v_C \cdot \left(\vec{\omega}_C \times \vec{v}_C \right)^2 + \left(\dot{\vec{\omega}}_C \times \vec{v}_C \right) \cdot \left(\vec{\omega}_C \times \vec{v}_C \right) \right) \tag{21}$$

$\Rightarrow$

$$dA = m \cdot \left(\dot{v}_C \cdot v_C - \vec{\omega}_C{}^2 \cdot \rho \cdot v_C + 2 \cdot v_C \cdot \left(\vec{\omega}_C \times \vec{v}_C \right)^2 + \vec{\omega}_C \cdot \dot{\vec{\omega}}_C \cdot \vec{v}_C{}^2 - \left(\dot{\vec{\omega}}_C \cdot \vec{v}_C \right) \cdot \left(\vec{\omega}_C \cdot \vec{v}_C \right) \right) \tag{22}$$

⇒

$$dA = \vec{F} \cdot \dot{\vec{R}} \cdot dt = m \cdot \left(\dot{\rho} \cdot \ddot{\rho} + \left(2 \cdot (\vec{\omega}_C \times \hat{\rho})^2 - \vec{\omega}_C{}^2\right) \cdot \rho \cdot \dot{\rho} + \rho^2 \cdot \left(\dot{\vec{\omega}}_C \times \hat{\rho}\right) \cdot (\vec{\omega}_C \times \hat{\rho})\right) \quad (23)$$

⇒

$$dA = \vec{F} \cdot \dot{\vec{R}} \cdot dt = m \cdot \left(\dot{\rho} \cdot \ddot{\rho} + \left(\vec{\omega}_C{}^2 - 2 \cdot (\vec{\omega}_C \cdot \hat{\rho})^2\right) \cdot \rho \cdot \dot{\rho} + \rho^2 \cdot \left(\dot{\vec{\omega}}_C \times \hat{\rho}\right) \cdot (\vec{\omega}_C \times \hat{\rho})\right) \quad (24)$$

⇒

$$dA = \vec{F} \cdot \dot{\vec{R}} \cdot dt = m \cdot \left(\left(\ddot{\rho} + \left(\vec{\omega}_C{}^2 - 2 \cdot (\vec{\omega}_C \cdot \hat{\rho})^2\right) \cdot \rho\right) \cdot \dot{\rho} + \rho^2 \cdot \left(\dot{\vec{\omega}}_C \times \hat{\rho}\right) \cdot (\vec{\omega}_C \times \hat{\rho})\right) \quad (25)$$

⇒

$$dA = \vec{F} \cdot \dot{\vec{R}} \cdot dt = m \cdot \left(\left(\ddot{\rho} + \left(\vec{\omega}_C{}^2 - 2 \cdot (\vec{\omega}_C \cdot \hat{\rho})^2\right) \cdot \rho\right) \cdot \dot{\rho} + \rho^2 \cdot \left(\dot{\vec{\omega}}_C \cdot \vec{\omega}_C - \left(\dot{\vec{\omega}}_C \cdot \hat{\rho}\right) \cdot (\vec{\omega}_C \cdot \hat{\rho})\right)\right) \quad (26)$$

As angular velocity is always perpendicular to the radius vector, i.e. $\vec{\omega}_C \perp \hat{\rho}$, we have:

$$W = \vec{F} \cdot \dot{\vec{R}} = m \cdot \left(\dot{\rho} \cdot \ddot{\rho} + \vec{\omega}_C{}^2 \cdot \rho \cdot \dot{\rho} + \left(\dot{\vec{\omega}}_C \cdot \vec{\omega}_C\right) \cdot \rho^2\right) \quad (27)$$

Therefore Coriolis force is not reactive or conservative one as frequently considered in literature, therefore it is real force fully capable to perform active work.

It is pertinent place to remind us that conservative or passive forces are the ones producing reactive power (usually forces perpendicular to the trajectory) and the active forces are fully capable to produce real work as they are not perpendicular to the trajectory.

Active power is scalar defined as:

$$dW_{active} = \vec{F} \cdot d\vec{\ell} \quad (28)$$

Reactive power is vector defined as:

$$d\vec{W}_{reactive} = \vec{F} \times d\vec{\ell} \quad (29)$$

This is pure evidence that this force is not cause of the declination of missiles or pendulums’ rotations due to the globe’s rotation, quite contrary the pendulum or the bullet declines from the globe plane due to absence of the Coriolis force that exists on the globe’s surface. Any pendulum is some sort of vibrating gyroscope and the gyroscopic effect causes pendulums to roate. Gyroscopic effect and Coriolis’ effect are quite different ones in their very essences; Coriolis Effect is planar or 2D effect while gyroscopic effect is duly spatial or 3D effect.

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